Gaussian Elimination

Gaussian elimination is used to solve the system of linear equations $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The method consists of combining the coefficient matrix A with the right hand side b to the ``augmented'' $(n, n + 1)$ matrix

$$[A\ b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}.$$

A sequence of elementary row operations is then applied to this matrix so as to transform the coefficient part to upper triangular form:

- multiply a row by a non-zero real number c,
- swap two rows,
- add c times one row to another one.

$[A\ b]$ will then have taken the following form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} & b'_n \end{bmatrix}$$

and the original equation is transformed to $Rx = c$ with an upper triangular matrix R, from which the unknowns x can be found by back substitution.

Assume we have transformed the first column, and we want to continue the elimination with the following matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ 0 & a'_{32} & \cdots & a'_{3n} & b'_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} & b'_n \end{bmatrix}.$$

To zero a'_{32} we want to divide the second row by the ``pivot'' $a'_{22}$, multiply it with $a'_{32}$ and subtract it from the third row. If the pivot is zero we have to swap two rows. This procedure frequently breaks down, not only for ill-conditioned matrices. Therefore, most programs perform ``partial pivoting'', i.e. they swap with the row that has the maximum absolute value of that column.

``Complete pivoting'', always putting the absolute biggest element of the whole matrix into the right position, implying reordering of rows and columns, is normally not necessary.

Another variant is *Gauss-Jordan elimination* , which is closely related to Gaussian elimination. With the same elementary operations it does not only zero the elements below the diagonal but also above. The resulting augmented matrix will then look like:

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 & b_1 \\ 0 & a'_{22} & \dots & 0 & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & a'_{nn} & b'_n \end{bmatrix}.$$

Therefore, back substitution is not necessary and the values of the unknowns can be computed directly. Not surprisingly, Gauss-Jordan elimination is slower than Gaussian elimination.

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